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Linear Algebra section 006

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Quiz #1

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

Problems: textbfProblems: Let

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

1. Give the definition of the characteristic polynomial and the characteristic equation of A. Compute then for A.
2. Deduce the eigenvalues of A and give their multiplicities.
3. Compute a basis for each eigenspaces and give their dimension.
4. Deduce if A is diagonalizable, and if so diagonalize A.
5. Give a basis for $Col(A)$ and $Row(A)$.

Solution:

1. The characteristic polynomial of A is $det(A - \lambda I)$ and the characteristic equation is $det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix}$$

$$\begin{aligned} det(A - \lambda I) &= (1-\lambda)(2-\lambda) - 12 = \\ &= 2 - 2\lambda - \lambda + \lambda^2 - 12 = -10 - 3\lambda + \lambda^2 = (\lambda - 5)(\lambda + 2) \end{aligned}$$

2. The eigenvalues of A are the roots of the characteristic polynomial thus they are 5 with multiplicity 1 and -2 with multiplicity 1.

3. The eigenspace of A corresponding to -2 is $Nul(A + 2I)$. So we need to find the dimension of the solution set of the equation $Ax = -2x$. For this we can row reduce the corresponding augmented matrix:

$$[A + 2I, 0] \sim \ \textit{Row reduce} \ \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We observe the reduced form of the augmented matrix and see that there is two free variable so that the dimension of the eigenspace of -2 is 1. The system corresponding to the reduced form of the augmented matrix is

$$\{ \; x_1 + x_2 = 0$$

The general form of an element of $Nul(A + 2I)$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

with x_2 scalar.
Thus $\{v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ is a basis for $Nul(A + 2I)$.
The eigenspace of A corresponding to 5 is $Nul(A - 5I)$. So we need to find the dimension of the solution set of the equation $Ax = 5x$. For this we can row reduce the corresponding augmented matrix:

$$[A - 5I, 0] \sim \; \textit{Row reduce} \; \sim \begin{pmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We observe the reduced form of the augmented matrix and see that there is one free variable so that the dimension of the eigenspace of -3 is 1 and equal to its multiplicity.
The system corresponding to the reduced form of the augmented matrix is

$$x_1 - 3/5x_2 = 0$$

The general form of an element of $Nul(A - 5I)$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/5x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3/5 \\ 1 \end{pmatrix}$$

with x_2 scalar.
Thus $\{v_2 = \begin{pmatrix} 3/5 \\ 1 \end{pmatrix}\}$ is a basis for $Nul(A - 5I)$.

4. Since A has distinct eigenvalues so we already know that A is diagonalizable. Putting

$$P = [v_1, v_2]$$

and

$$D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$$

where the eigenvalues in D correspond to v_1, v_2 respectively. So that,

$$A = PDP^{-1}$$

5. $Col(A) = Span\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\}$. Clearly, $\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\}$ is linearly independent since these two vectors are not multiple. So, $\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\}$ is a basis of $Col(A)$.
   $Row(A) = Span\{(1,3),(4,2)\}$. Clearly, $\{(1,3),(4,2)\}$ is linearly independent since these two vectors are not multiple of each other. So $\{(1,3),(4,2)\}$ is a basis for $Row(A)$.